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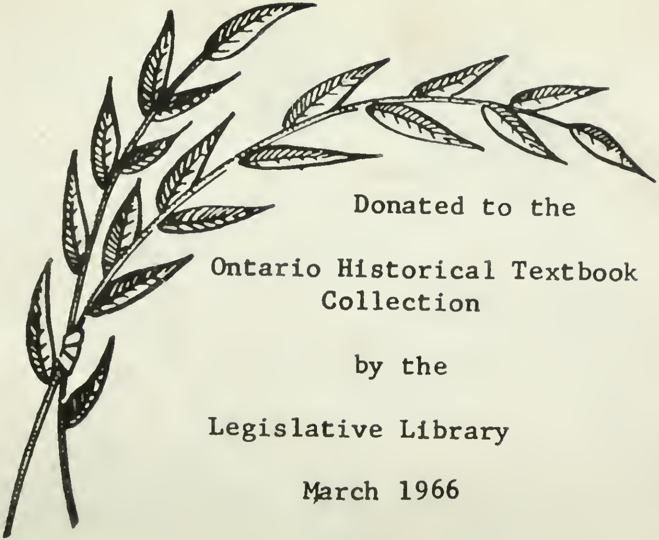

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ELEMENTS

OF

GEOMETRY,

FOR THE

USE OF SCHOOLS.

PARTS FIRST AND SECOND.

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PREFACE.

This Treatise contains the First and Second Parts of Clairaut's Elements of Geometry, translated from the French—a work hitherto neglected from its want of scientific demonstration. It contains a view of the principal truths of Geometry in the order in which they were discovered, and on that account seemed best suited for the present purpose. The Author was aware that many who might have derived advantages from an acquaintance with such a useful branch of science, and perhaps been allured to the study of more scientific works, were discouraged by the dry and unpromising aspect it presents when taught synthetically, and found the necessity of adopting a more simple plan for an elementary book.

Among those whose circumstances do not permit a proper training of their intellectual faculties, there are many whose usefulness and comfort might be increased by a *degree of intellectual education* suited to their occupations; and if there is any branch of knowledge which can adapt itself to an indefinite variety of circumstances in the lower ranks of a civilized nation, it is Geometry,—there is scarcely a theorem which is not applicable to some useful purpose.

The following Treatise, written by an eminent Mathematician, has been strongly recommended by Lacroix, for the purpose of opening the minds of children, and

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preparing them, under judicious teachers, for higher studies; and is now published, in the hope that it may assist many in the practical purposes of humble industry.

The Author, in his Preface, observes:—"It will be objected, that in certain parts of this work I rely too much on the testimony of the eyes, and neglect the strictness of demonstration. I beg that whoever makes the objection, will consider that I pass thus slightly only over propositions whose truth may be discovered by the smallest degree of attention. I proceed in that manner, especially at the beginning, (where we meet with the greatest number of such propositions,) because I have observed that persons who have any talents for Geometry, are gratified by such opportunities of using their own ingenuity, while on the contrary, they turn away, almost in anger, when they find themselves oppressed under a multitude of demonstrations, which, properly speaking, are useless."

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ELEMENTS OF GEOMETRY.

PART FIRST.

MEANS WHICH APPEAR THE MOST NATURAL TO HAVE BEEN EMPLOYED IN MEASURING LAND.

Lengths and distances must have been the first things measured.

I.

To measure any length whatever, a kind of natural Geometry will suggest the comparison of a known measure with the length which it is desired to ascertain.

II.

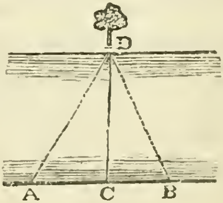
In regard to distance, it is clear that in order to measure that which lies between two points, it is necessary to draw a *straight* line from the one to the other, on which to apply successively the known measure; for since all other lines make some circuit, they must be longer than the right line which makes none.

A right line is the shortest that can be drawn from one point to another; and consequently the measure of the distance between two points.

III.

As we are frequently obliged to measure the distance between two points, so it also happens that we want to know the distance from a certain point to a line. A person, for instance, placed at D, on the bank of a river, wishes to ascertain how far it is to the other bank A B. It is clear, in this case, that to have this measure, it is necessary to take the shortest of all the right lines D A, D B, &c. which may be drawn from the point D, to the right line A B. Now, it is easy to observe that the line wanted is D C, which must not lean to the line A B more in the direction of the right hand than of the left. On this line, therefore, which is called the perpendicular, we must move our known measure, in order to ascertain the distance D C, from the point D, to the right line A B. But it is evident that if we are to apply the measure to D C, that the line must be previously drawn. We are therefore under the necessity of finding a method of drawing perpendiculars.

A line falling upon another without inclination to either side, is a perpendicular to that line.



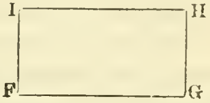
IV.

Such perpendiculars are wanted on many other occasions. We know, for instance, that the regularity of such figures as A B C D, F G H I, called rectangles, and consisting of four sides, perpendicular to each other, has induced men to give that shape to their houses, their rooms, &c.

Rectangles are figures having four sides perpendicular to each other.

The first of these figures A B C D, whose four sides are equal, is called a *square;* the second F G H I, in which the opposite sides only are equal, is called a *rectangle.*

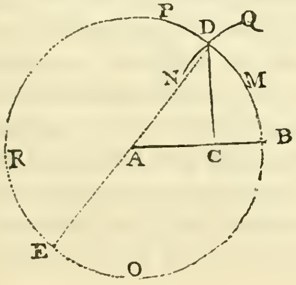
A square is a rectangle which has its four sides equal.



V.

In the various operations which require that a perpendicular be drawn, it is proposed either to let it fall upon a line from a point out of it, or to raise it from a point in the line itself.

Method of drawing a perpendicular.



Suppose that on the point C, in the line A B, we wish to raise the line C D, perpendicular to A B. To effect this it will be necessary that the line in question be without inclination either towards A or B.

Supposing then that C is at the same distance from A and B, and that the right line C D, does not lean either way, it is clear that every point of it will be at an equal distance from A and B. All therefore we have to do is to find any one point D, whose distance from A and B shall be equal; for in that case if we draw a right line C D, through this point and C, this line will be the perpendicular required.

The point D, might be found by repeated trials, but this method would leave the mind unsatisfied; for a method which gives no light, affords no gratification.

The following will be found satisfactory;—Take a common measure, a rope line for instance, or a pair of compasses with a certain opening, according as you may be going to operate either on the ground or on paper.

When you have determined the measure, fix, on the point A, either one extremity of the rope, or one of the points of the compasses; and by making the other point or extremity move, it will draw the arc P D M. After which, without changing your measure, fix one extremity or point on B, and describe the arc Q D N, which cutting the other arc in D, will give you the point required just where the two arcs cross each other.

For since the point D, belongs equally to the two arcs P D M, Q D N, described by means of a common measure, its distance from the point A, must be equal to its distance from the point B. Accordingly C D, will not incline either towards A or B: Therefore this line will be perpendicular to the line A B.

If the point C, is not equally distant from A and B, we must take two other points A and B, at equal distances from C, and employ them as A and B, to describe the arcs P D M. Q D N.

VI.

If one of the traces of the point of the compasses or extremity of the rope (see the preceding figure) such as P B, were continued by O E and R, till it returned into itself at P, where it began; the whole trace would be called the circumference of a circle, or simply a circle.

A circle is the space contained within the whole line, traced by one point of a pair of compasses turning round the other.

If only a part P D M, of a circumference is drawn, this part is named an arc of the circle.

The fixed point A, is called the centre of that arc, or of the whole circle.

The centre is the point on which the leg of the compasses is fixed.

The distance A D, is called the radius.

The radius is a line equal to the interval between the points of the compasses.

Every line which, like D A E, passes through the centre A, and is terminated by the circumference on both sides, is called a diameter. It

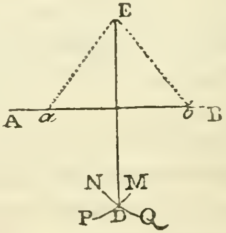
A diameter is twice the radius.

is evident that such a line is twice the radius; hence the radius is sometimes called semi-diameter.

VII.

Method of letting fall a perpendicular.

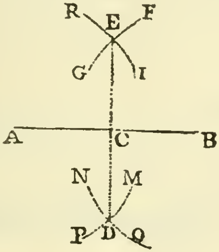
The method of raising a perpendicular upon a line A B, furnished that of letting it fall from any point E, given out of the line: for if we fix the extremity of a string or one of the two points of the compasses on E, and mark two places *a*, *b*, at the same distance E *b*, on the line A B, we may find, by the preceding article, another point D, at equal distances from *a* and *b*. A line drawn by the points D and E, will have both extremities equally distant from *a* and *b*, that is, will not be more inclined to *a* than to *b*, and consequently will be perpendicular to A B.



VIII.

From the preceding operation follows the solution of a new problem. Suppose we want to divide a right line A B, into two equal parts. From the points A and B, as centres, and with any opening of the compasses, describe the arcs R E I, G E F; then from the same centres and with the same or any other opening, describe the arcs P D M, Q D N. The line E D, which joins the two points of E and D, where the arcs cross each other, will divide A B into two equal parts, at the point C.

To divide a line into two equal parts.

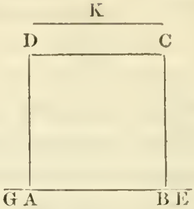


IX.

Having found the method of drawing perpendiculars, nothing is easier than, by means of that method, to construct those figures which are called squares, and which are mentioned in Art. IV. It is evident that in order to make a square A B C D, the sides of which shall be equal to the given line K, we must take on the right line G E, a portion A B, equal to K, and raise from its points A and B, (Art. V.) the

To construct a square, the side being given.

perpendiculars A D, B C, both equal to K, joining them by D C.

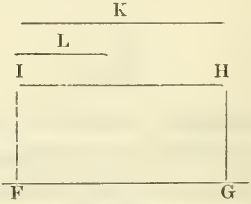


If A B were an inch, the square A D C B would be called a square inch; or, if a foot, it would be called a square foot.

X.

To construct a rectangle, the length and height being given.

If we wished to form a rectangle F G H I, with the length K, and the height L, we should make F G equal to K, and then raise the perpendiculars F I and G H, both equal to L, joining them by H I.



XI.

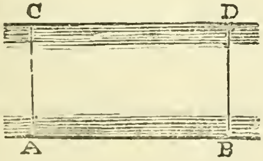
Parallels are right lines, whose distance from each other is always the same.

In the construction of works such as canals, streets, &c. it is necessary to draw parallel lines—that is, lines so situated that their distance from each other may be measured at every point by perpendiculars of the same length.

To draw such lines nothing seems more natural than the method of forming rectangles. Let A B, for instance, be one of the sides of a canal, to which it is intended to give the breadth C A; or to state the problem more geometrically and generally, suppose it is desired to draw through the point C, the line C D, parallel to A B. Take any point B, on the line A B, and proceed in the same manner as if, the base A B, being given, it were required to make the rectangle A B C D, with the height A C. In this case if the lines C D, A B, were produced, without limitation, they would always continue parallel, or which is the same thing, they would never meet.

The method of forming rectangles naturally suggests a method of drawing such line.

Through a given point to draw a parallel to another line.



XII.

As owing to their regularity rectangular figures are frequently used, there are many cases in which it is important to know their dimensions. We may want, for instance, to know how much carpeting is required for a room, or how many acres are contained within a rectangular building-ground, &c.

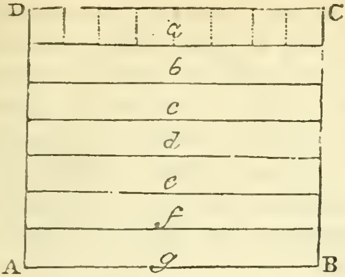
We easily perceive that the easiest and most natural way of determining this is to use a common measure, which being applied several times upon the surface to be measured, shall cover it completely:—a method reducible to that which has already been used to determine the length of lines.

Now it is evident that the measure of a surface must itself be a surface; for instance a square yard, square foot, &c. Thus to measure a rectangle is to find the number of square yards, or square feet, &c. contained in its surface or area.

Let us make this clearer by an example. Suppose that the given rectangle A B C D, is seven feet high upon a base of eight feet. This rectangle may be considered as divided into seven slips, *a, b, c, d, e, f, g,* each containing eight square feet. Its whole surface will therefore be seven times eight square feet, or 56 square feet.

The measure of a rectangle is the product of its base by its altitude.

By recollecting the first elements of arithmetical calculation we may here observe a perfect analogy between the operation of measuring a rectangle, and that of common multiplication, which consists in taking one number as many times as there are units in another. For if we multiply the number of yards, feet, &c. which represent the length or base, we shall have the number of square yards, square feet, &c. contained in the whole area.

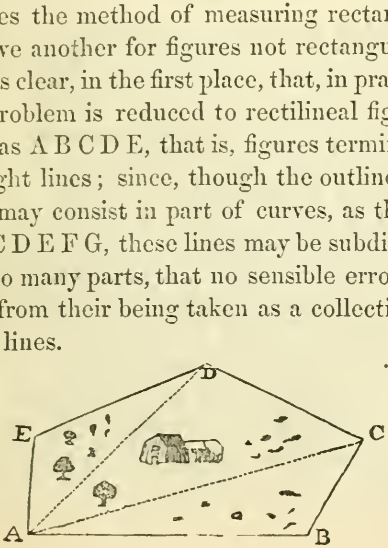


XIII.

The figures to be measured are not always as regular as rectangles. But as nevertheless we must ascertain their dimensions—for instance, when some work has been raised on an irregular plot of ground, or to know the acres contained in a field—it was found necessary, besides the method of measuring rectangles, to have another for figures not rectangular.

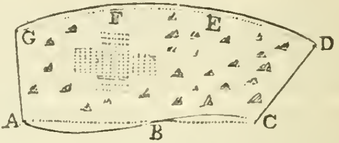
It is clear, in the first place, that, in practice, the problem is reduced to rectilineal figures, such as A B C D E, that is, figures terminated by right lines; since, though the outline of a field may consist in part of curves, as that of A B C D E F G, these lines may be subdivided into so many parts, that no sensible error will arise from their being taken as a collection of right lines.

Rectilineal figures are those which are terminated by right lines.



A rectilineal triangle is a figure terminated by three right lines.

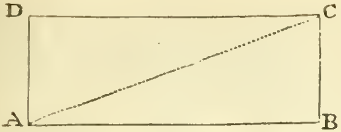
Having laid down this principle, it is easy to see, that in spite of the great variety of rectilineal figures, all may be measured by dividing them into figures of three sides, called triangles. This may be done in a very simple and easy manner, if from any point A, on the sides of the figure A B C D, we draw the right lines A C, A D, &c., to the points C D, &c.



XIV.

All that we have to do is to find the manner of measuring the triangles we have drawn. But to find out what we do not know, the surest way is to try whether there is something in what we do know, which is connected with the object of the inquiry. We have already seen that any rectangle A B C D, is equal to the product of its base A B, by its height C B. We may also easily perceive that this figure transversely divided by the line A C, which is called a diagonal, resolves itself into two equal triangles; from which we infer, that each of them must be equal to half the product of their base A B or D C, by their height C B or D A.

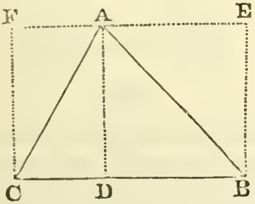
The diagonal of a rectangle is a line which divides it into two equal triangles.



It is true that the triangles which occur in practice have seldom if ever two of their sides perpendicular to each other, as the triangles A B C, A D C, which are called right-angled triangles; but they may be reduced to that kind of triangles.

Right angled triangles are those which have two of their sides perpendicular to one another.

For let the perpendicular A D, be made to fall from the vertex A, on the base B C, of any triangle A B C; the triangle A B C, will be divided into two right-angled triangles A B D and A D C.



Resuming then what has been said, it is evident that, since the two triangles A B D, A D C, would be halves of the rectangles A E B D, A D C F, the triangle A B C, will also be the half the rectangle E B C F, having B C for its base, and A D for its height. But since the surface of the rectangle E B C F, is equal to the product of the height E B, or A D, by the base B C, the triangle A B C, will have as its

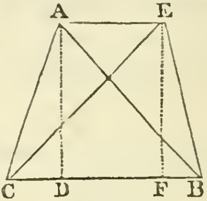
A triangle is the half of a rectangle which has the same base and height.

measure, the half of the product of the base B C, by the perpendicular A D, that is, the altitude of the triangle.

XV.

The areas of triangles which have the same altitude and base are equal.

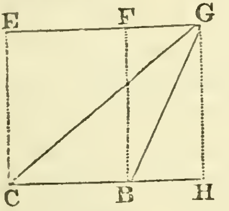
Since in measuring the area or surface of triangles, we only employ the base and the altitude, without considering the length of their sides, we may deduce this proposition or theorem—any triangles E C B, A C B, which have a common base C B, and whose altitude, E F, A D, are equal, have also equal areas.



XVI.

To make still clearer the principle on which triangles are measured, we have thought it necessary to choose as a base such a side, that a perpendicular drawn from the vertex must fall upon it; which may be done whenever we have to measure land. But as, in comparing triangles which have the same base, the perpendiculars drawn from the vertices may fall out of the triangle, as in the annexed figure, it seems necessary to ascertain whether such triangles as B C G, are in the same case, that is, whether they are always the half of any rectangle E C B F, which has the perpendicular G H, as the measure of its altitude.

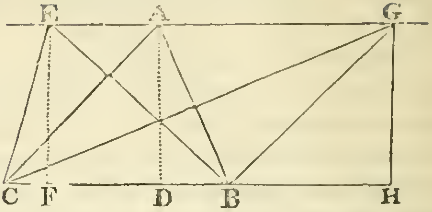
Of this we may easily be convinced by observing that the triangle C G H, made up of the two triangles C G B, G B H, is the half of the rectangle E C H G, which is itself made up of the two rectangles E C B F, F B H G, so that the sum of the two triangles C G B, G B H, is the half of the rectangle E C H G. But the triangle G B H, is the half of the rectangle F B H G, therefore the triangle B C G, is the half of the other rectangle E C B F, which has B C as its base, and G H as its altitude.



XVII.

Triangles which have the same base, and are within the same parallels, are equal in superficies.

The proposition which has been demonstrated in the three preceding articles, may be stated generally in the following terms—any triangles E B C, A B C, G B C, which have the same base B C. and lie between the same parallels E A G, C B H, are equal; that is, when their vertices are in the same right line E A G, parallel to the right line C B : for then their altitudes measured by the perpendiculars E F, A D, G H, are equal.

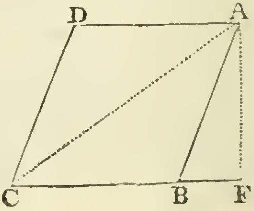


XVIII.

Parallelograms are four-sided figures, whose opposite sides are parallel.

Among the various rectilineal figures, which may be measured by the preceding method, there are some which approach rectangles in regularity: such for instance as A B C D, terminated by four sides, each of which is parallel to the one opposite. These figures are called parallelograms; and are the easiest to measure, with the exception of rectangles. For, divide the parallelogram A B C D, into two triangles A B C, A C D, these two triangles are equal. And as each of these triangles is equal to half the product of the altitude A F, and the base B C, the parallelogram must be equal to the whole of that product.

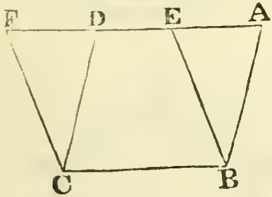
Parallelograms are measured by multiplying their altitude by their base.

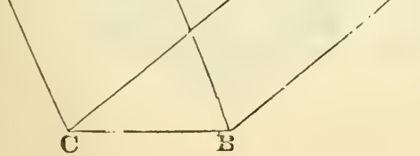


XIX.

It follows from this, that all parallelograms A B C D, E B C F, having a common base and lying between the same parallels, must be equal. This may be seen in both the adjoining figures, independently of what has been said, by observing that the parallelogram A B C D, becomes the parallelogram E B C F, by adding to it the triangle D C F, and taking from the whole figure the triangle A B E. But since the two triangles D C F, A B E, are supposed to be equal, it is evident that the parallelogram A B C D is not altered in dimensions by becoming E B C F. Now to be certain that these triangles are equal, we have only to observe, that A B and C D, being parallels as well as B E and C F, the triangle A B E may be considered as the triangle D C F made to glide on its base till the point A coincides with D, and E with F.

Parallelograms which stand upon the same base, and are between the same parallels, are equal.



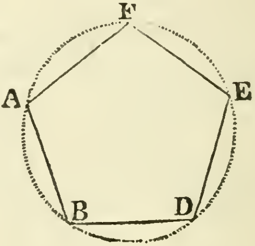


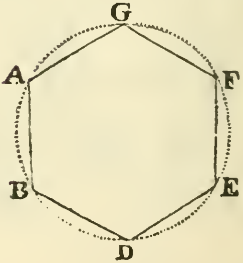
XX.

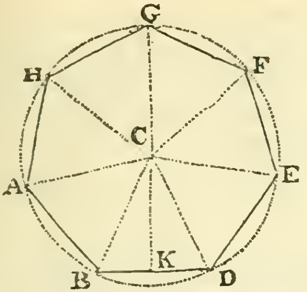
Regular polygons are rectilineal figures terminated by sides all equal and equally inclined to each other.

There are still other rectilineal figures whose measurement is easy, and which are called regular Polygons, or figures terminated by equal sides, with the same mutual inclination. Such are the figures A B D E F, A B D E F G, A B D E F G H.

As the symmetrical form of these figures is in very common use, I think it necessary, before giving the method of measuring them, to give that of drawing them.







XXI.

Method of describing a polygon.

Describe a circle and divide its circumference, by trial with a pair of compasses, into as many equal parts as the polygon is to have sides. Draw then the lines A B, B D, D E, through the points A, B, D, E, &c., which mark the divisions of the circumference previously made, and the polygon will be formed. It will be a pentagon, hexagon, heptagon, octagon, enneagon, &c., according to the number of its sides.

XXII.

Measure or the surface of a regular polygon.

In measuring a regular polygon, we might employ the method given (Article XIII.) for all rectilineal figures. But it is clear that the shortest way is to divide the polygon (see the third figure, Article XX.) into equal triangles having their vertices in C, the centre. For if we take, for instance, one of the triangles, C B D, and draw the perpendicular C K

A perpendicular drawn from the centre of a polygon to one of its sides, is called the apothem.

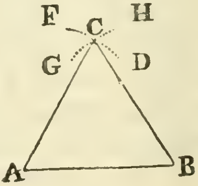
upon its base, the area of the triangle will be equal to the product of the base B D by half C K: and this taken as many times as there are sides in the polygon, will be the area of the whole figure.

XXIII.

An equilateral triangle is that which has its three sides equal to one another.

If the circumference of the circle were divided only into three parts, the result would be a triangle commonly called equilateral: if into four equal parts, it would be a square. But these figures (the simplest of all polygons) may be easily drawn without dividing a circle: as we have shown in regard to the square, (Article IX.) As to the equilateral triangle, it will be easily seen that, in order to describe it on a given base A B, we should take the distance A B on the compasses, and draw the arcs D C F, G C H, from the points A and B, as centres, joining the point of intersection of the arcs with the points A and B.

Method of describing it.



XXIV.

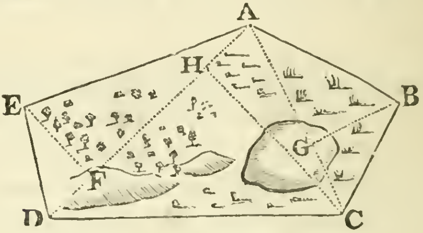
To the method of describing geometrically equilateral triangles and squares, we might add that of constructing a pentagon. But as we write exclusively for beginners, who without the knowledge of Algebra would not be able to perceive the train of reasoning by which the method of drawing such a figure was found, we shall not explain that operation for the present.

When I say that Algebra is necessary for the construction of polygons of more than five sides, I do not mean to speak of such as have 6, 12, 24, 48, &c. or 8, 16, 32, 64, &c. sides, which may be easily constructed by methods which elementary Geometry teaches, as will be seen at the end of this First Part.

XXV.

I return to the measurement of land, and observe that there are cases where the ground presents insuperable obstacles to the methods hitherto prescribed.

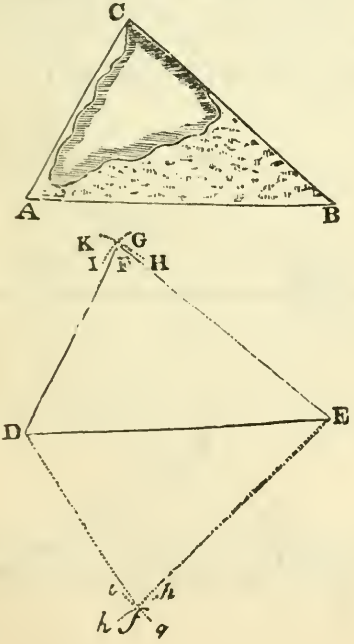
Suppose that we have to measure a field, a plot of ground, &c. such as A B C D E. According to what has been said, it would be necessary to divide A B C D E into such triangles as A B C, A C D, A D E, which we should have to measure by drawing the perpendiculars E F, C H, B G. Suppose, however, that in the space A B C D E, a rising ground, a wood, a pond, oppose our drawing the lines required, how shall we get over this difficulty? The method which seems most obvious is that of taking a portion of level ground on which we may work without impediment, and there describe triangles equal and similar to A C D, A B C, &c. Let us see how we might contrive to draw those new triangles.



XXVI.

The three sides of a triangle being given, to describe another triangle equal to it.

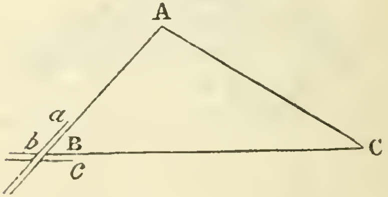
Let us begin by supposing that the obstacle is found within the triangle A B C, whose sides are known, and that we wish to describe another triangle equal and similar on the chosen ground. In the first place we must draw the line D E equal to the side A B: then taking a string of the length B C, and fixing one of its ends at E, we must describe the arc I F G of which the string is the radius. By means of another string equal to A C with one end fixed on D, we shall draw the arc K F H, intersecting the other in F. Lastly, we shall draw the lines D F, and F E, which will complete the triangle D E F, equal and similar to the proposed triangle A B C. This is evident; for since the sides D F, E F, which meet at the point E, are equal, respectively, to the sides A C, B C, which meet at the point C, and since the base D E has been made equal to A B, it is impossible that the position of the lines D F, and E F on D E should be different from that of the lines A C, B C on A B. It is true that we might have drawn the lines D *f*, E *f*, on the other side of the base D E; but the triangle would be the same, though in an inverted position.

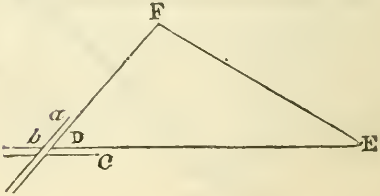


XXVII.

If only two of the three sides of the triangle A B C could be measured, for instance, A B, B C, it is clear that this would not be enough to construct another triangle equal and similar to A B C. For though we might take D E equal to B C, and D F equal to B A, we should not be able to fix the relative position of the lines. To remove this difficulty, this simple expedient presents itself, viz. to give to D F the same inclination towards D E as A B has towards B C; or to use the language of Geometry, to give the angle F D E the same opening as that of the angle A B C.

An angle is the inclination of one line towards another.





XXVIII.

Method of making an angle equal to another.

To perform this an instrument represented by *a b c* is used. (See figures in the preceding article.) It consists of two rules capable of turning on the point *b*, which are laid on the sides A B and B C. In this state they must make the same angle as the sides A B and B C. If we therefore apply the rule *b c* to the base D E, so that the centre *b* shall be upon the point D, the opening of the instrument remaining the same, the ruler *a b* will give the position of the line D F, which, with the line D E, will make the angle F D E equal to the angle A B C. As the line D F is supposed to have been made of the same length as B A, nothing more is required than to draw through F and E the right line F E, in order to have the triangle F E D equal and similar to the triangle A B C. This simple method proceeds on the evident principle, that a triangle is determined by the length and the opening of two of its sides, or, (what is the same) that a triangle is equal to another when two of their sides are respectively equal, and the angle which they inclose has the same opening.

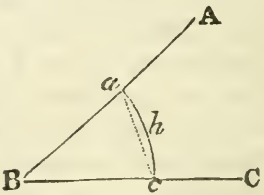
A triangle is determined when two sides and the angle which they contain are given.

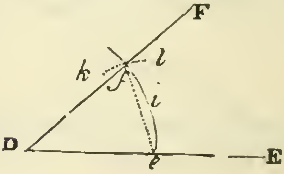
XXIX.

There is still another method of making the angle F D E equal to A B C.

The chord of an arc is the right line terminated by the two extremities of the arc.

From the centre B, and with any distance B *a*, describe the arc *a h c*, and from the centre D, and with the same interval, draw the arc *e i f*. This being done, you have only to find a point *f*, which shall have the same position in the arc *e i f* as *a* has on the arc *c h a*. But it is easy to find the point *f* by means of the right line *a c*, which, according to the established definition, is called the chord of the arc *a h c*. For if from the centre *e*, and with an interval equal to *a c*, you describe the arc *l f k*, the intersection of the two arcs *e i f*, *l f k*, will give you the required point *f*.—Draw now through D and *f*, the line D *f* F, and the angle F D E will be equal to the angle A B C. This is evident (from Art. XXVI.) since the triangles B *a c*, D *f e*, will be entirely equal and similar in all their parts.

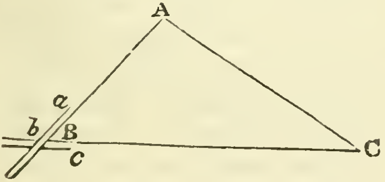


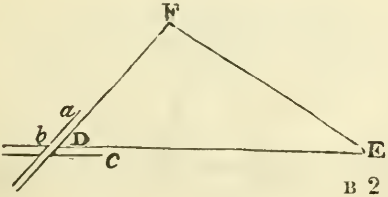


XXX

Two angles and one side determine a triangle.

If we wish to make the triangle F D E equal to the triangle A B C, and yet could measure but one of the sides, B C for instance, we must avail ourselves of the angles A B C and A C B. Having made D E equal to B C, let us place the lines F D and F E in such a relative position as to make with D E the same angles as A B and A C with B C, and by their meeting they will make a triangle equal and like to A B C. The principle supposed in this operation is so simple, that it need not be demonstrated.





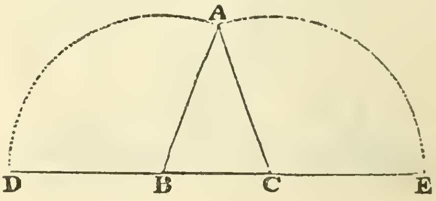
XXXI.

An isosceles triangle has two of its sides equal.

If of the three sides of the triangle A B C we could measure no other but the base B C, and yet we knew that this triangle was *isosceles*, that is, had the two sides A B, A C equal to one another; it is evident that the measurement of either angle A B C or A C B would be enough for our purpose, since the other angle would be equal.

The reason of this will appear, if we imagine what would happen if the two sides A B, A C, of the triangle A B C were lying on B D, and C E (which are prolongations of the base B C,) and afterwards raised, in order to join their extremities at the point A. For (supposing them to move at the same rate), the equality of their lengths would not allow them to move through different portions of space; but they would meet, having performed the same distance. Their inclination, therefore, to the base B C, after their junction, would be equal; and consequently the angle A C B would be equal to the angle A B C

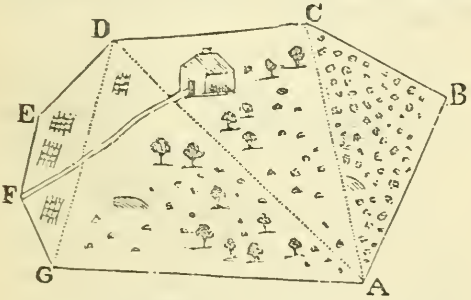
The angles formed by the equal sides with the base, are equal to one another.



XXXII.

To return to the measure of ground, it will be observed that whatever be the obstacles which may be met with, the preceding method enables us without difficulty to transfer to an open space, all the triangles into which the ground to be measured may have been divided.

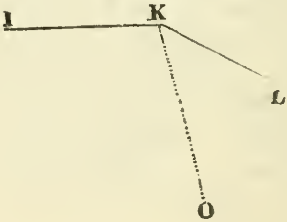
Suppose, for instance, that we wished to measure a wood whose limits formed the figure A B C D E F G.



First we should construct a triangle equal to A B C, which might be done without penetrating into the area of the triangle, by measuring the two sides A B, B C, and the angle C B A contained by them.

Having described this triangle we should have the value of the angle B C A, and the length of A C, and since we could measure the exterior side C D, we should have the sides D C, C A, of the triangle C A D. As to the angle D C A, we might find it by first drawing the angle I K L, equal to the angle D C B, and then the angle L K O, equal to the angle B C A, which would give you the remaining angle I K O, equal to the required angle D C A.

After determining the triangle A C D by means of the two sides D C and C A and the inclosed angle D C A, the triangle D A G and the rest of the figure might be ascertained by the same method.

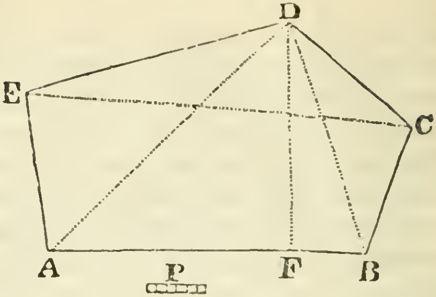


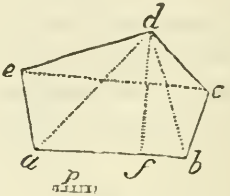
XXXIII.

The method we have given for the measurement of ground on which there are impediments to the drawing of lines, is often exposed to great difficulties in practice. It is not indeed easy to find a sufficient extent of free and level ground on which to construct triangles of the same dimensions as those which we wish to measure. And even if we could command such ground, the great length of the sides of the triangles might render the operations very difficult. To draw a perpendicular upon a line from a point at the distance, not greater than 1,000 yards, would be an operation of great difficulty, and perhaps impracticable. It is therefore important to have a method of superseding such operations.

Such a method is almost obvious. It will easily occur to any one, that the figure A B C D E, which we have to measure, may be represented by a smaller figure *a b c d e*, similar to the former, in which the side *a b*, for instance, may be 100 inches long, when the side A B is 100 yards; the side *b c* 45 inches, if B C is 45 yards; and hence to conclude, that as the area of the reduced figure *a b c d e* is 60,000 square inches, that of the figure A B C D E must be 60,000 square yards.

But we must find, in the first place, in what the similarity of the two figures does consist.





XXXIV.

What constitutes the similarity of figures.

Little reflection is needed to perceive that two figures, A B C D E, *a b c d e*, will be similar, if the angles A, B, C, D, E, of the greater are equal to the angles *a*, *b*, *c*, *d*, *e*, of the less; and the sides *a b*, *b c*, *c d*, &c. of the less contain the part or measure *p*, as often as the sides A B, B C, C D, &c. of the greater contain the part or measure P.

XXXV.

Geometricians express this latter condition by saying that the sides A B, B C, C D, &c. should be proportioned to the sides *a b*, *b c*, *c d*, &c.; that is, that the side A B should contain *a b*, in the same manner as B C contains *b c*, &c., or that the side A B should be as great, compared to *a b*, as B C is compared to *b c*: in other words, that there be the same ratio, or relation, between A B and *a b*, as there is between B C and *b c*, &c.; or lastly, that A B be to *a b* as B C is to *b c*, &c. All these expressions mean the same thing; but it is necessary to make them familiar to the mind in order to understand the language of Geometry.

XXXVI.

Method of drawing a figure similar to another

Having seen what the similarity of two figures consists in, let us try to find the most natural method of drawing a figure similar to another. For this purpose, let us imagine a draughtsman who wishes to copy a figure on a smaller scale than the original.

First, taking *a b*, (see the same figure,) to represent the base A B of the figure A B C D E, which he intends to copy, he inclines the sides *a e*, *b c*, with respect to *a b*, in the same manner that A E and B C are in respect to A B, and makes the length of *a e*, *b c*, such, that they shall be to *a b* as the length of A E, B C is to that of A B, that is, if A E is, for instance, the half of A B, he will make *a e* equal to half *a b*. The same will be done in determining the length of *b c*, in respect to B C.

Having in this manner determined the points *e* and *c*, he will draw the lines *e d*, *c d*, inclining them in the same manner as E D, C D are to E A and C B, and producing them till they meet at *d*; when the figure *a b c d e* will be complete

XXXVII.

If we now reflect on the construction of this figure, we shall see that it all depends on the equality of the angles E, A, B, C, and the angles *e, a, b, c,* and on the proportionality of the sides E A, A B, B C, with the sides *e a, a b, b c;* so that the figure is completed without having taken pains to make the angle *d* equal to the angle D, nor the sides *e d, c d* proportional to the sides E D, C D. This might produce a doubt whether the angle *d* were really equal to the angle D, or the sides *ed, cd,* proportional to the sides E D, C D, and, consequently, whether the figure *a b c d e* were entirely similar to A B C D E. Experience alone would, however, be enough to remove the doubt; but a slight reflection will convince us that the respective equality of the four angles E, A, B, C, to *e, a, b, c,* and the proportionality of the three sides E A, A B, B C, and *e a, a b, b c,* must produce the equality of the angles D and *d*, and the proportionality of the sides E D, C D, with *e d, c d.*

But to remove all doubt, we will show that all the conditions which the similarity of two figures requires, are necessarily dependent on each other.

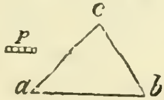
This will appear by the examination of triangles, which being the simplest figures, enter into the composition of all other figures. Such an examination will lead us to the knowledge of the properties and uses of similar figures.

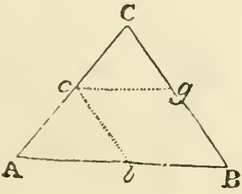
XXXVIII.

If two angles of a triangle are equal to two angles of another triangle, the remaining angle of the one will be equal to the remaining angle of the other.

Let us suppose that in constructing the triangle *a b c* upon the base *a b*, we are exclusively concerned in making the angles *c a b*, *c b a*, equal to the angles C A B, C B A. We may convince ourselves, in the first place, that the remaining angle *a c b* will be equal to the remaining angle A C B. For let the triangle *a c b* be applied upon the triangle A B C so that the point *a* fall upon the point A, *a b* upon the line A B, *a c* upon A C; it is clear that *c b* will be parallel to C B, because, if the side *c b* were produced, it could not meet the side C B unless these two lines were unequally inclined towards A B. But as this would imply that the angles *c b a*, C B A, were not equal, it is directly against the supposition.

As from the equality of the angles *c b a*, C B A, it follows that the lines *c b*, C B, must be parallel, so from the parallelism of the lines, it will follow that the angles A *c b*, A C B, must be equal; which was to be proved.



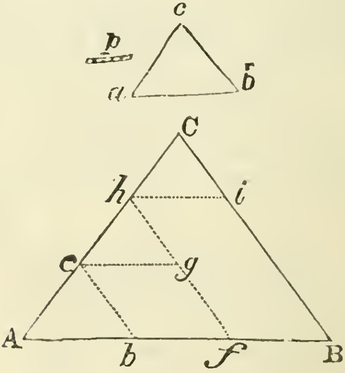


XXXIX.

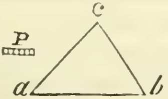
Now let us prove (see the same figures,) that the corresponding sides in the triangles *acb* and A C B, that is, the sides which contain the same angles, are proportional to each other.

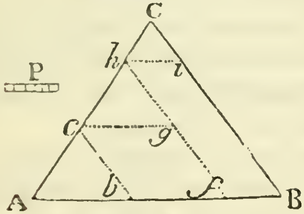
Two triangles whose angles are respectively equal, have their sides proportional to each other.

To give distinctness to our ideas, let us suppose that *a b* (see figures Art. XXXVIII.) is the half of A B. In this case we shall have to prove that *a c* is also the half of A C, and *b c* the half of B C. Let *a c b*, as in the preceding article, keep the position A *c b*. If we draw *c g* parallel to A B, it is clear that it will be equal to *b* B or A *b*, and that *g* B will also be equal to *c b*. But as the angles *cg*C, and C*cg*, are evidently equal to the angles *c b* A and *c* A *b*, the triangle C *c g* will be equal to the triangle *c* A *b* (Art. XXX.) C *c* therefore will be equal to A *c*, and C *g* equal to *c b* or *g* B. A *c* or *a c* therefore will be the half of A C, and *c b* the half of C B.



If *a b* were contained three, four, or any number of times in A B, it would be equally easy to demonstrate that *a c* would be contained the same number of times in A C, and *c b* in C B. For if from the points of division *b*, *f*. in the base A B. we drew *b c*, *f h*, &c. parallel to B C, we might place along B C three, four, &c. triangles A *c b*. *c h g*, *h* C *i*. &c. equal to the triangle *a c b*. But if we suppose that *a b*. instead of being contained a certain number of times in A B, measured it with a fraction. twice for instance, and a half more, we may prove in the same manner that *a c* would be equally contained in A B, and *b c* in B C.





For, when by means of the parallels *b c*, *f h*, we had placed along A C the two triangles A *c b*, *c h g*, equal to *a c b*, there would remain a space between the parallels *h f* and C B. where we might place a triangle C *h i*, whose sides would be the half of the sides *c* A *b*. This is evident, for by our supposition *f* B is the half of A *b*, and the base *h i* of the triangle C *h i* must be equal to *f* B, as being contained between the parallels *h f*, C B. In general, when two triangles A B C, *a b c*, have the same angles, the sides of these *similar* triangles are proportional, that is, the sides A B, B C, A C, of one of the triangles A B C, contained the same number of times the measure P, as the sides *a b*, *b c*, *a c*, of the other triangle *a b c*, contain the measure *p*. By P we understand the foot, yard, &c. or, in general, the scale used in the construction of A B C; and *p*, that employed in the construction of *a b c*.

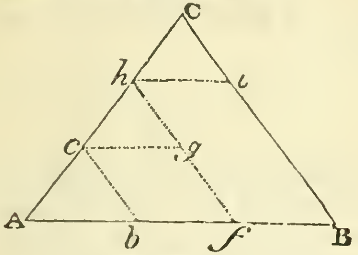
XL.

To divide a line into any number of equal parts.

From the proportion which we have just demonstrated, we derive the solution of a problem of frequent use in practice.

Let it be required to divide a line into a given number of equal parts. This might be done by repeated trials; but without the certainty afforded by Geometry.

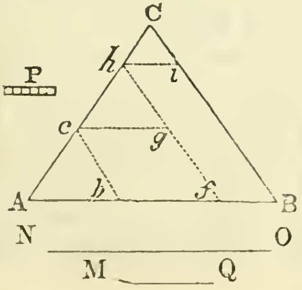
Let us suppose, for instance, that we have to divide A B into three equal parts. Draw an indefinite line A C, making any angle with A B. On A C, with any opening of the compasses, take the three equal parts A *c*, *c h*, *h* C, and from the point C, draw C B. Draw the lines *c b*, *h f*, so as to be parallels to B C. The points *b f*, where *c b*, *h f*, meet A B, will divide it into three equal parts. This has been proved in the preceding article.



XLI.

What a fourth proportional line to three others is, and how it may be found.

If we wished to divide a line into a fractional number of parts, such as two and a half, or three and one fourth, &c., or, generally, if we wished to divide the line A B at the point *b*, so that A B should be to A *b* as the line N O to M Q; it is clear that the solution of this problem would result from Art. XXXIX.; that is, it would be necessary to draw from A a right line, and take upon it A *c* and A C, equal respectively to M Q and N O, and then draw *c b* parallel to C B. The point required would be *b*.

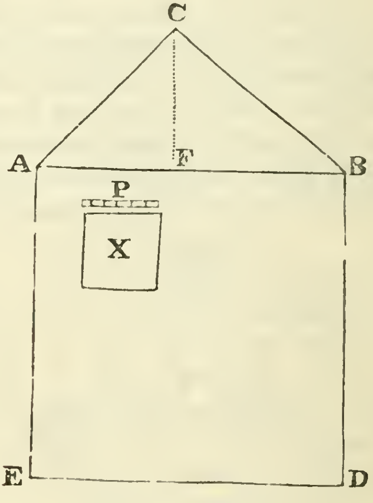


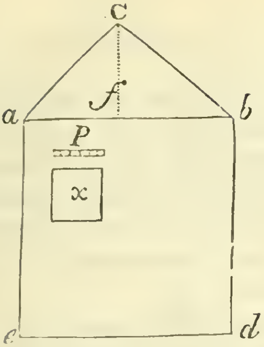
Geometricians express the problem which we have solved, in the following words: To find a fourth proportional to the lines N O, M Q, A B.

XLII.

The altitudes of two similar triangles are proportional to their sides.

It is evident that the two similar triangles A B C, *a b c*, will have not only their sides proportional, but also that the perpendiculars C F, *c f*, drawn from the vertices C, *c*, to the bases A B, *a b*, will follow the proportion of the sides. This may be so easily demonstrated from what goes before, that we will not stop to prove it.





XLIII.

As to the areas of the similar triangles A B C, $a\,b\,c$, it is clear that that of the first will contain as many squares X formed upon the measure P, as the area of the other will contain squares x, constructed on the measure p. For as C F and A B will contain (by the preceding article) as many parts P, as $c\,f$ and $a\,b$ will have parts p, the half of the product of C F and A B, which is the measure of A B C, (Art. XIV.) will be the same number as the half of the product of $c\,f$ and $a\,b$, the measure of $a\,b\,c$. But there will be this difference, that C F and A B being reckoned in parts P, their product will be reckoned in squares X; and as $c\,f$, $a\,b$, are reckoned in parts p, their product will be made up of squares x.

XLIV.

The areas of similar triangles are to each other as the squares of their homologous sides.

What we have just said on the measure of similar triangles, proves a proportion which in the elements of Geometry, is expressed in the following manner. Similar triangles A B C, *a b c*, are to each other as the squares A B D E, *a b d e*, of their corresponding or homologous sides A B, *a b*.

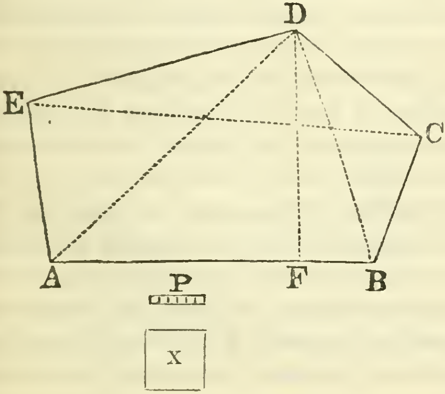
The demonstration contained in the preceding article leads directly to this inference. For since the square A B D E, contains as many X's as *a b d e* contains *x*'s, it is clear that the two numbers of squares X, which express the ratio (or proportional size) of the triangle A B C, to the square A B D E, must be the same as the number of squares *x*, which express the ratio of the triangle *a b c*, to the square *a b d e;* in other words, the triangle A B C is to the square A B D E, as the triangle *a b c*, to the square *a b d e*.

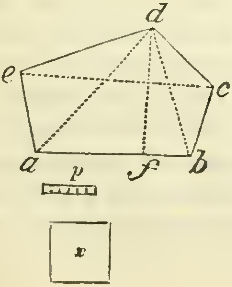
Hence it follows that if, for instance, the side A B were double of the side *a b*, the triangle A C B would be quadruple of the triangle *a c b:* if A B were triple of *a b*, the triangle A C B would be nine times greater than the triangle *a c b*, &c.; for A B cannot be double of *a b*, without the square A B D E being quadruple of the square *a b d e*, &c.

XLV.

To proceed from triangles to other figures, let us suppose that to each of the similar triangles A B D, *a b d*, two other triangles A D E, B D C, and *a d e*, *b d c*, are joined, also respectively similar. In the figures resulting from them all A B C D E, *a b c d e*, we may observe:—

Properties of similar figures deduced from those of triangles.





1. That the angles A, B, C, D, E, will be equal to the angles a, b, c, d, e, which is evident; since all will be corresponding angles of similar triangles, or angles made up of such corresponding angles.

2. That the ratio of the homologous or corresponding sides D E, $d\,e$, B C, $b\,c$, &c. of the figures A B C D E, $a\,b\,c\,d\,e$, will necessarily be the same; that is, if the measure P is contained a certain number of times in the base A B, and p the same number of times in $a\,b$, P and p will also be contained an equal number of times in any two homologous sides whatever, D E and $d\,e$; for, owing to the similarity of the triangles A B D, $a\,b\,d$, the number of P measures contained in A D will be equal to that contained in $a\,d$. But, if we consider these sides as the bases of the similar triangles A D E, $a\,d\,e$, the number of measures P contained in D E will equal the number of measures p contained in the side $d\,e$.

3. It will appear, moreover, that if in the two similar figures we were to draw corresponding lines, such as C E, $c\,e$, or the perpendiculars D F, $d\,f$, &c., these lines would be to each other in the same ratio as the homologous sides of the two figures.

The two figures A B C D E, $a\,b\,c\,d\,e$, will, therefore, be perfectly similar.

XLVI.

The figure *a b c d e* being thus described in perfect similarity to A B C D E, it is evident that if we wished to construct another figure equal and similar to *a b c d e,* and therefore similar also to A B C D E, it would be useless to measure all the sides and all the angles of *a b c d e;* and that it would suffice, for instance, to take the three sides *a b, e a, b c,* and the four angles *e, a, b, c,* in order to draw an equal figure *a b c d e,* which would be similar to A B C D E. And here we have a complete demonstration of what was only suggested as a conjecture in Art. XXXVII. But we may proceed farther: for it is clear that there will always be various ways of combining the number of angles and lines which must be measured in any figure, in order to construct another which shall be proportional to it. But we will not fatigue the reader with a more minute detail.

XLVII.

The areas of similar figures are to each other as the squares of their homologous sides.

It may be demonstrated by reasonings similar to those of Art. XLIII. that the number of squares X contained in the figure A B C D E, is the same as that of square *x,* contained in the figure *a b c d e;* and, therefore, the areas of similar figures are to each other as the squares of their homologous sides.

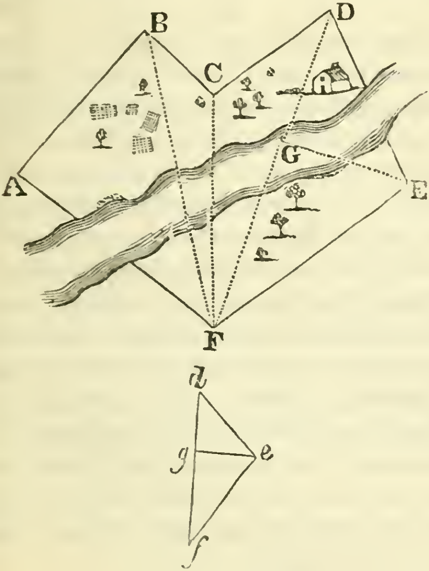
XLVIII.

Similar figures are distinguished only by the scale on which they are constructed.

All that has been said on similar figures may be reduced to this single principle, that the only difference between similar figures arises from the scale used in their construction.

XLIX.

But the better to understand the use which should be made of similar triangles and reduced figures, for the purpose of having the measure of grounds on which it would be difficult to operate; let us suppose that A B C D E F, represents the outline of a park, of a reservoir, &c. of which we wish to know the extent. Here we should first measure one of the sides of the figure, for instance F E, and having ascertained its length in yards, furlongs, or any other measure, we must trace on paper, or any other plane surface, a line *f e*, containing as many parts of any scale whatever, as F E contains yards, or furlongs, &c.: and making the angles *d e f*, *d f e*, equal to the angles D E F, D F E, we shall have the triangle *e d f*. Upon the side *d f*, draw the perpendicular *e g*. Let us then measure the lines *d f*, *e g*, in parts of the adopted scale; and as many of these parts as we shall find in them, so many yards, or perches, or furlongs, &c. will be contained in D F and E G. By multiplying, therefore, D F by the half of E G, we shall have the value of the triangle E D F. By the same method we shall be able to measure each of the other triangles D C F, B C F, A B F; the sum of which will be equal to the whole figure.



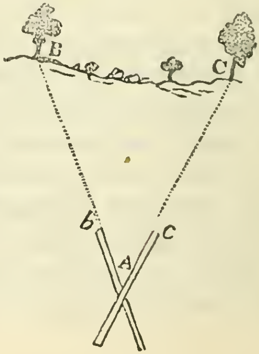
L.

Method of measuring the distance of an inaccessible point.

It often happens in practice, that it is necessary to measure the distance of the point F, on which we stand, from another point which we cannot reach. But the solution of this problem has already been given in the preceding article. For since to measure D F, we require only the similarity of the triangles *d e f*, &c. D E F, it is evident, that, if we measure any base E F, and are able to see D from the points F and E, the problem will be solved, that is, we shall have the length of the line F D.

LI.

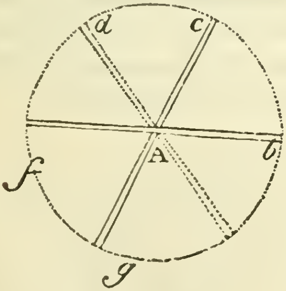
The use of such instruments as *b* A *c*, (mentioned already Art. XXVIII.) consisting of two rulers, joined at A, so that they may turn round that point, is liable to many objections. For in transferring the angle, the aperture may be altered; and it frequently happens that the shape, which for the sake of convenience we are obliged to give the instrument, prevents our applying it to the surface on which we are about to draw the reduced figure. Add to this, that every new angle, taken in this manner, requires the application of the instrument on the paper, and that the only way in which we can compare two angles, is that of placing one over the other, without, however, being able to know exactly either their relative or their absolute magnitude.



LII.

Hence the necessity of finding an accurate measure for the angles, as we have already one for the distances. This was accomplished without much difficulty. For suppose that A *b*, being kept immoveable, we place upon it the side A *c*, and then make it turn round A: it is clear, that, had we attached to the extremity *c*, of the moveable ruler A *c*, a pen or pencil which should mark the course of the point *c*, the trace, which would be an arc of a circle, would give the exact measure of the angle formed by the two rulers at every aperture A *b*, A *c*; that is, owing to the uniform curvature of the circle, a double, triple, or quadruple aperture of *c* A *b*, will have an arc double, triple, quadruple, of the arc *c b*.

An angle is measured by the arc of a circle which its sides intercept.



LIII.

If, therefore, we suppose that the circumference $b\,c\,d\,f\,g$, described by the complete revolution of the point c, is divided into any number of equal parts, the number of such parts contained in the arc intercepted by the lines A c, A b, will measure the opening of those lines; or the angle c A b, which they make.

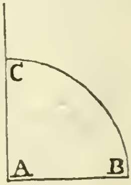
The circle is divided into 360 degrees, each degree into 60 minutes, &c.

Geometricians have agreed to divide the circle into 360 parts, which are called degrees; each degree into 60 minutes; each minute into 60 seconds. Thus an angle b A c, for instance, will be of 70 degrees, and 20 minutes, if the arc $b\,c$, which measures it, contains 70 of the 360 parts of the circle, and 20 sixtieth parts of a degree besides.

LIV.

A right angle contains 90 degrees, and its sides are perpendicular to each other.

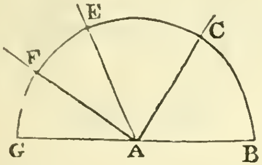
Hence it follows that an angle C A B of 90 degrees, commonly called a right angle, is one whose sides A C and A B, cut B C, that is, one-fourth of the circumference. These sides are perpendicular to each other.



LV.

Every angle which contains less than 90 degrees is called *acute*. Such are the angles CAB, FAG, EAG.

An acute angle is smaller than a right one.



LVI.

On the contrary, an angle FAB, which contains more than 90 degrees, is called *obtuse*.

An obtuse angle is greater than a right angle.

LVII.

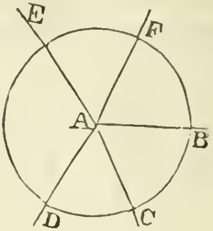
It is evident that all the angles, such as GAF, FAE, EAC, CAB, which may be made on one side of a right line GB, having the same vertex A, are, when taken together, equal to 180 degrees, or two right angles, whose measure is a semicircle, or half the cirumference.

The sum of all the angles which may be made on the same side of a right line, with the same vertex, is equal to 180 degrees.

LVIII.

In the same manner, the sum of all the angles EAF, FAB, BAC, CAD, DAE, &c., which may be made round the point A, as a common vertex, are equal to 360 degrees, or four right angles, measured by the whole circumference BCDEF.

All the angles which may be made round one point, are equal together to four right ones.



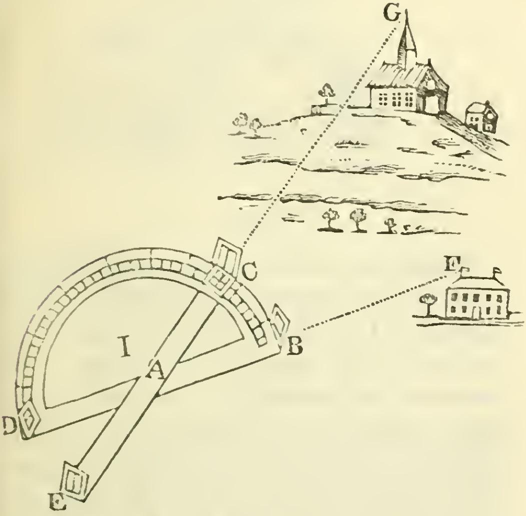
LIX.

Having found that angles are measured by arcs of circles, let us see how we are to ascertain the number of degrees contained by an angle.

For this purpose we use what is called a semicircle. This instrument is made up of two equal rulers E A C, D A B, crossing each other at A, and terminating in the sights B, D, E, C. One of these rulers, E C, turns upon A: the other, D B, is immoveable, and is the diameter of D C B; a semicircle divided into 180 degrees, &c.

Suppose then that we want to find the angle formed by two right lines drawn from the place where we stand, to any two objects F, G. Place the fixed ruler D A B, so that the eye applied at D, may see one of the two objects, F, through the two sights D and B. Then, without moving the instrument, direct the other ruler, so that the eye placed at E may see the other object G, through the sights E, C. The edge of the moveable ruler will mark on the graduated circle the number of degrees, minutes, &c., contained by the given angle G A F.

Use of the instrument called a semicircle, in measuring angles.

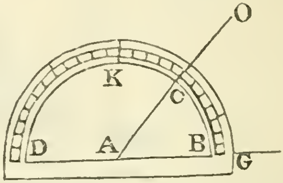


LX.

When we wish to draw on paper an angle of a given number of degrees, we use the instrument K, which is also divided into 180 degrees. It is called the *Reporter*. Placing the centre A on the point of the angle which

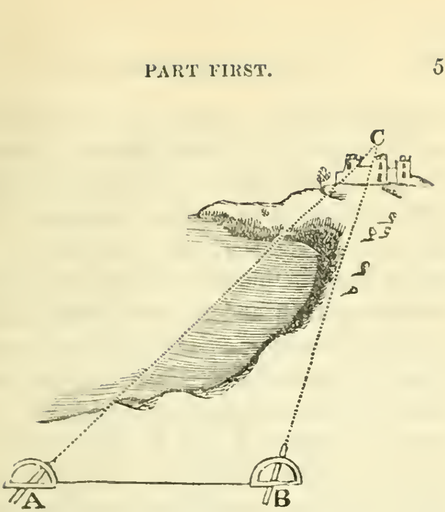
Use of the reporting semicircle in drawing an angle of a given number of degrees.

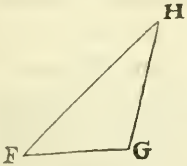
we wish to draw, and the edge A B, on the line A G, one of the sides of the intended angle, mark the point C, where you find the number of degrees required. Through this point and the centre A draw the line A C O, which, with the other, will make the angle O A G, containing the degrees required.



LXI.

Suppose now, that having taken on paper any base F G, we wish to construct upon it a triangle F G H, similar to the triangle A B C, taken on the ground. By means of the semicircle or theodolite, let us ascertain the degrees contained in the angles C A B, C B A, and let us transfer these angles to the paper by means of the *Reporter*. We shall have the angles H F G, and H G F, respectively equal to C A B, and C B A. And because the point H, where the sides F H and G H meet, must necessarily be determined by the operation, as well as the angle F H G, we shall have the triangle F G H entirely similar to the triangle A B C.

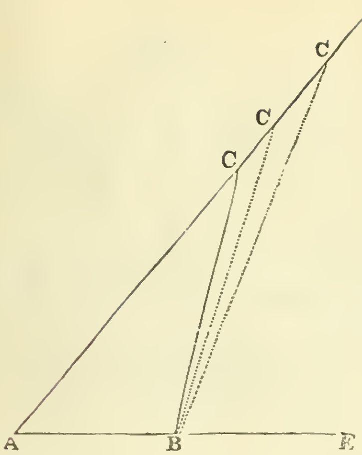




LXII.

As it is of consequence, in practice, that the angles be exactly measured, we should not trust entirely to the most accurate instruments. We must endeavour to find a method of verifying these measurements, in order to correct them if necessary. This method is, however, as easy as it is simple. Let us return to the triangle A B C. It is clear that the extent of the angle C depends on that of the angles A and B; for if we either enlarge or diminish these angles, the proportion of the lines A C, B C, must change, as well as the size of the angle C, which is formed by them.—Now, if this angle depends on the size of the angles A and B, it may be presumed that the number of degrees contained by the angles A and B, must determine the number of degrees contained by the angle C. Thus the size of that angle may be made a proof of the operations by which we determine the angles A and B. For if we find that the angle C contains the number of degrees which corresponds to the magnitude of the angles A and B, there can be no doubt that the measurement was correct.

To find how from the size of the angles A and B we may find that of the angle C, let us examine what effect would be produced on this angle by the approach or deviation of the lines A C, B C. Let us suppose, for instance, that B C, moving on the point B, recedes from A B, and approaches B E. It is evident, that by this motion the angle B would be made wider and wider; and the angle C would, on the contrary, become closer and closer. This might lead us to suspect that the diminution of the angle C would be equal to the increase of the angle B; so that the sum of the three angles A, B, C, would always be the same, whatever the inclination of the lines A C, B C, towards the line A E, might be.

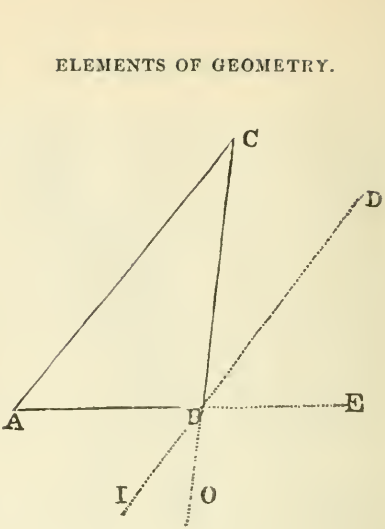


LXIII.

Now this presumed inference involves its own demonstration. For, draw I D parallel to A C. It will be evident, in the first place, that the angles A C B and C B D (called alternate angles) will be equal; for, as the lines A C and I B are parallel, their inclination towards the line C B O must be the same; that is, the angle I B O will be equal to the angle A C B. But the angle I B O must be equal to the angle C B D, because the right line I D cannot be more inclined to C O on one side than on the other; therefore the angle D B C, which is equal to the angle I B O, is also equal to its alternate angle A C B.

Alternate angles are the reversed angles formed on each side of a right line, which falls upon two parallel lines.

Alternate angles are equal.



LXIV.

It will be seen in the next place, that on account of the parallelism of C A and D B, the angle C A E is equal to the angle D B E. We have it, therefore, in our power to consider the three angles as joined by their vertices at B; and to infer that the three angles C A B, A C B, and C B A, which are equal to the angles D B E, C B D, C B A (the sum of which, according to Art. LVII. is equal to two right angles), are also equal to two right angles. What we have observed here being applicable to all triangles, we may lay it down as a universal property of triangles that the sum of their angles is invariably the same, and equal to two right angles, or 180 degrees.

The sum of the three angles of a triangle is equal to two right angles.

LXV.

If, therefore, we wish to ascertain the value of the third angle of a triangle, when we have the measure of the other two, we have only to subtract the sum of these two angles from 180, and the remainder will be the value of the third angle. This property of triangles affords an easy method of verifying the measurements of angles. Many other applications of this principle will appear as we proceed. We shall here mention the more direct inferences.

LXVI.

A triangle cannot have more than one right angle; and still less, more than one obtuse angle.

LXVII.

If one of the angles of a triangle be a right angle, the sum of the two other angles must be equal to a right angle.

These two propositions are too clear to require demonstration.

LXVIII.

The exterior angle of a triangle is equal to the two interior angles opposite to it.

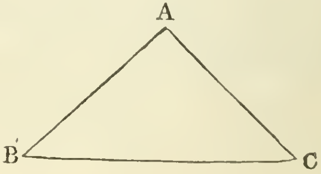
If one of the sides A B, of the triangle A B C be produced, the exterior angle C B E will be equal to the two interior, and opposite angles, B C A, C A B. For whether to the angle C B A we add either the two angles B C A and C A B, or the angle C B E, the sum will be equal to 180 degrees, or two right angles. (Article LXIV.)

LXIX.

If one of the angles of an isosceles triangle A B C, be given, the two others are known.

For, let the vertical angle A be known or given, it is clear, that by subtracting the number of degrees it contains, from 180, which is the sum of the three angles, each of the other two B, C, will measure half the remainder.

But, let the known angle be one of the angles B, C, the double of its measure subtracted from 180, will be the measure of the vertical angle A.



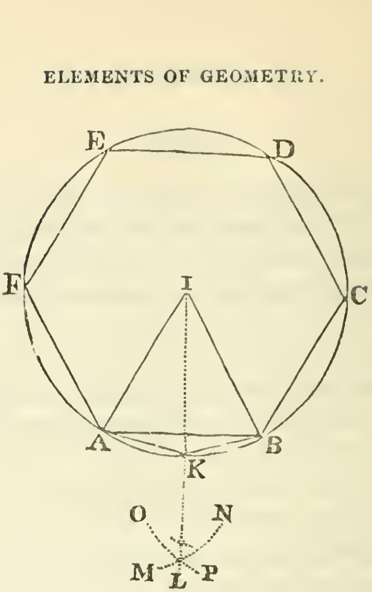
LXX.

As an equilateral triangle is one which, placed on any side as its base, will be an isosceles triangle, it is evident that all its angles are equal, and each measure 60 degrees.

Each angle of an equilateral triangle measures 60 degrees.

LXXI.

Hence, we may easily give the description of the hexagon, or polygon of six sides, which we promised in Art. XXIV. For, a line which shall divide the circumference into six equal parts, must be the chord of an arc of 60 degrees, the sixth part of 360. the number of degrees which measures the circumference. Supposing therefore that A B is such a chord, if, from the centre I, we draw the radii A I and I B. to the extremities A and B, the angle A I B will be one of 60 degrees; and since its two sides A I, and I B, are equal, the triangle A I B will be an isosceles triangle. And as the vertical angle is of 60 degrees, each of the other two angles will also be of 60 degrees, which is the half of 120. The triangle A I B will consequently be equilateral (Art. LXX.) A B must therefore be equal to a radius of the circle. Hence it is clear that in order to describe a hexagon, we must take the length of the radius on the compasses, and carry it six times over the circumference, which will give us the six sides of the hexagon



LXXII.

Having described the hexagon A B C D E F, it will be easy to describe a dodecagon or polygon of twelve sides.

The half of an angle at the centre of a hexagon, is one of the angles at the centre of a dodecagon.

Divide the arc A K B, or the angle A I B, into two equal parts, and A K, the chord of half the arc A K B, will be one of the sides of the dodecagon.

LXXIII.

To divide an angle into two equal parts.

To divide the arc A K B into two equal arcs A K and K B, the same operation will be necessary as if we wanted to divide the chord A B into two equal portions. From the points A and B as centres, and with any distance whatever, draw the arcs M L N, O L P, and through the point L, where the arcs intersect each other, and the centre I, draw the line L I, which will divide the arc A K B, and the chord A B into two equal parts.

LXXIV.

To describe polygons of 24, of 48, &c sides.

By following the above method, and dividing the arc A K into two equal arcs, the chord of either will be a side of a polygon of 24 sides. In the same manner we may have polygons of 48, 96, 192, &c. sides.

LXXV.

To describe an octagon.

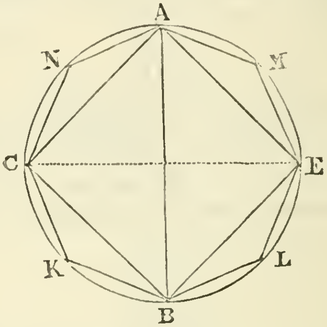
To describe an octagon, that is, a polygon of eight sides, we should begin by inscribing a square on the circle. This will be done, if, having drawn the diameters A I B, and C I E, crossing each other at right angles, we join their extremities by the lines A C, C B, B E, A E.

For, owing to the regularity of the circle, and the equality of the four angles formed by the perpendiculars A I B, C I E, the four sides A C, C B, B E, E A, are necessarily equal, and equally inclined to each other, which are the properties of a square.

When we have the square thus drawn, let us divide, as above, the arcs C K B, B L E,

&c. into two equal parts. This will give us the octagon C K B L E M A N.

By dividing again the arcs C K, K B, &c. into 2, 4, 8, &c. equal parts, we shall have polygons of 16, 32, 64, &c. sides.



END OF THE FIRST PART.

ELEMENTS OF GEOMETRY.

PART SECOND.

GEOMETRICAL METHOD OF COMPARING RECTILINEAL FIGURES.

Whoever shall have paid some attention to what has been hitherto said, in order to show how men arrived at the knowledge required to measure lands, must have observed that the relative position of lines could not fail to fix the attention for its own sake, and independently of the utility which it might afford in practice. It is indeed probable, that such observations as these engaged the early Geometricians to carry on their researches: for it is not want only that stimulates mankind; curiosity also is a fertile source of discovery.

There was another motive besides, to proceed in the path of Geometrical speculation. The love of accurate methods is so natural to the human mind, that it cannot be satisfied without them.

When therefore it was observed that scales and semicircles gave, in a great number of cases, only approximated values of lines and angles, many were moved to invent methods to supply the imperfection of those instruments.

We shall now resume the consideration of rectilineal figures; but in the operations which we are about to perform, in order to discover their exact relations, we shall use the rule and compasses only.

It frequently happens that we want to collect, in one figure, a certain number of figures which are similar to it; or that we wish to decompose a figure into several others of the same kind. This may be done by operating first on rectangles, since all rectilineal figures are only assemblages of triangles, and since every triangle is the half of a rectangle of the same altitude and base.

I.

That we may compare rectangles, we should know how to change any rectangle whatever, into another containing the same superficies, but with a different altitude. For as soon as we have reduced two rectangles to two others of the same altitude, they will differ only as to their bases; and that will be the greater rectangle that has the greater base; and it will contain the smaller exactly as its base contains the base of the smaller: a truth which is generally expressed thus:—Two rectangles which have the same altitude are in the ratio of their bases.

Two rectangles which have the same altitude are in the ratio of their bases.

II.

To add such rectangles to each other, nothing is required but to join them by their equal sides.

III.

If we wish to subtract the smaller from the greater, we shall find no difficulty in cutting off from the greater a portion equal to the smaller.

IV.

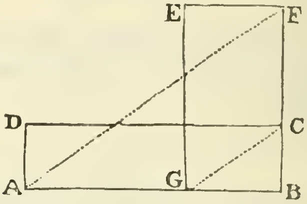
To divide a rectangle into a definite number of equal rectangles, we have only to divide the base into a like number of equal parts, and to raise perpendiculars on the points of division.

V.

Let it now be proposed to change the rectangle A B C D, into another which shall have the same surface, but with the altitude B F. Here we should remark, that as the area or surface of the rectangle must be the product of its altitude by its base, the new rectangle (the altitude of which is to be

Method of changing one rectangle into another of a given altitude.

greater than B C) must necessarily have a base less than A B; in other words, if B F is, for instance, twice as long as B C, the new base B G must be only one-half of A B.



If B F were three times B C, B G should be only one-third of A B.

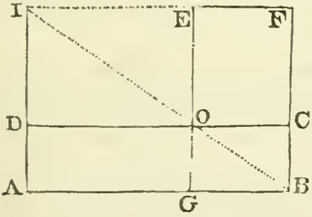
It may easily be seen, that if B F, instead of containing B C an exact number of times, contained it a certain number of times and a fraction, such as twice and one-third, the rectangle B F E G could not be equal to the rectangle A B C D, unless its base B G should be likewise contained twice and one-third in the base A B. In general, it will not be difficult to see that if the two rectangles A B C D, B F E G, are to be equal, the base B G of the one must be contained in the base A B of the other, as the altitude B C is contained in the altitude B F.

All, therefore, that we have to do, is to divide the line A B in such a manner, that A B shall be to G B as B F to B C. This (according to Art. XLI. Part First,) will be done by drawing the line F A, and, from the point C, the line G C parallel to F A.

VI.

Another method of changing one rectangle into another with a given altitude.

To change the rectangle A B C D into another rectangle B F E G, with a given altitude B F, we may employ a method which, though not so natural as the preceding, is nevertheless more convenient. Having produced (that is, prolonged) A D till it meets in I, the right line F E I, drawn from the point F parallel to A B, draw the diagonal B I, and through the point O, where it will cross the side D C, draw G O E parallel to F B. The rectangle B F E G will be equal to the rectangle A B C D.



To prove this assertion we need only show that, if from the rectangles A B C D, B F E G, we take away the portion O C B G, the remaining rectangles A D O G and E O C F will be equal.

This will be evident if we consider the equality of the two triangles I B F, I B A; for if from these equal quantities we take off equal quantities, the remainder will be equal. But the triangle I A B will become the rectangle A D O G, if we take away the two triangles I D O, O G B; and the triangle I B F will become the rectangle E O C F, by withdrawing the triangles I E O, O B C, which are respectively equal to the other two; therefore the two triangles A D O G, E O C F, (the remainders of the two triangles) will be equal to one another.

VII.

This second method of changing one rectangle into another, confirms the principle supposed in the first method; a principle which might appear to be founded upon mere inference.

If two rectangles are equal, the base of the one is to the base of the other, as the altitude of the second to the altitude of the first.

To satisfy the condition of the problem which required the equality of the two rectangles A B C D, B F E G, (see Fig. in Art. V.) we found it necessary to suppose that A B ought to be to B G as B F is to B C. We are now able to make it more manifest by means of the preceding article.

For, the triangles I A B and O G B, (Fig. in Art. VI.) being clearly similar to each other, (see Art. XXXIV. and Art. XXXV. Part First.) the base A B of the greater, will be to the base

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G B of the less, as the altitude I A to the altitude O G, or as the lines B F, B C, which are equal to them. Whence, A B is to G B as B F is to B C; which agrees with what was said at the beginning of Art. V.

VIII.

If four lines are such that the first is to the second as the third to the fourth, the rectangle contained by the first and the fourth will be equal to one contained by the second and third.

By the method which we have pursued in demonstrating that, since the two rectangles A B C D, B F E G, are equal, the altitude B F is to the altitude B C as the base A B is to the base B G, we may also show that if four lines B F, B C, A B, B G, shall be such that the first is to the second as the third to the fourth; a rectangle with the first of those lines for base and the fourth for altitude, must be equal to another rectangle with the second and third as altitude and base.

IX.

Four quantities, the first of which is to the second as the third is to the fourth, are said to form a proportion.

When four quantities, in the manner of lines B F, B C, A B, B G, are such, that the first is to the second as the third to the fourth, these quantities are said to form a *proportion*, or to have the same ratio.* Thus, 6, 9, 18, 27, form a proportion; because 6 is contained in 9 in the same manner as 18 is contained in 27. The same occurs with 15, 25, 75, 125, &c.

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* Ratio is the relation which one quantity (whether number or extension) bears to another in respect of magnitude,—the comparison being made by considering what multiple, part or parts, one is of the other.—See Art. XXXV., Part First.

X.

The first and fourth terms of a proportion are called the *extremes:* the second and third the *means.*

The first and fourth quantities in a proportion are called the *extremes;* the second and third the *means.*

If we use the terms just defined, to express the proportions contained in Articles VII. and VIII., they will be as follows:—

XI.

In every proportion the product of the extremes is equal to the product of the means.

When four quantities make a proportion, the product of the extremes is equal to the product of the means.

XII.

If the product of the extremes is equal to the product of the means, the four quantities form a proportion.

If four quantities are such, that the product of the extremes is equal to the product of the means, those four quantities form a proportion.

XIII.

On these principles is grounded the Rule of Three.

The two preceding articles should be particularly noticed, as being of great use, and especially as containing the principle which demonstrates the arithmetical operation called the Rule of Three. To give an idea of that rule, the most effectual way will be an example.

Suppose that 24 workmen have done 30 yards of some work in a certain time; we wish to know how much of the same work will 64 workmen complete in the same time.

It is evident that to answer this question we must find a number which shall be to 64, in the same ratio, as 30 is to 24. But according to what we have observed, that number will be such that its product by 24 will be equal to the product of 30 by 64. Now, the product of 30 by 64 is 1920. The number therefore which we seek will be that which multiplied by 24 will give for product 1920. The slightest acquaintance with Arithmetic is sufficient to perceive that such a number must be the quotient of 1920 divided by 24, that is 80.

Method to find the fourth term of a proportion when the first three are given.

Generally, to find the fourth term of a proportion, when the first three are given, we should divide the product of the second and third, by the first term. The quotient will be the fourth.

XIV.

So simple a case as that which we have chosen, will probably not be sufficient to show the necessity of the method just stated. Common sense might indeed enable us to find the number required. For we may easily observe that 30 exceeds 24 by one-fourth; and consequently the number of workmen required must also exceed 64 by one-fourth; and thus we might find that the number is 80. But there are cases in which we could not find the relation of the first two numbers without employing a long time in the search. Suppose, for instance, that we wanted a fourth proportional to the three numbers 259, 407, 483.

The preceding method enables us to find it, by the multiplication of 483 by 407, and by the division of 196581, the product by 259. The quotient 759, is the fourth sum we wanted.

Were it not for this method we must have groped our way by means of trials. We might indeed have found that 148, the excess of 407 above 259, contains four-seventh parts of 259; and therefore it would be necessary to add to 483, the number of 276, which contains likewise 4 of its seventh parts. But the generality and certainty of the method above explained spares us all such round-about calculations, which indeed in many cases would be perfectly unavailing.

XV.

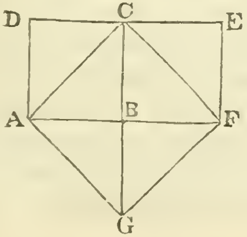
The addition of two squares will be performed in the same manner as that of two rectangles, since squares are only rectangles whose altitude and base are equal. We may therefore change one of the squares (say, the smaller) into a rectangle with the altitude of the greater square, and in this manner we shall make a rectangle out of the two squares.

We might indeed give the altitude of the smaller square to the greater, or any altitude whatever. But another problem must have presented itself, while Geometricians were endeavouring to reduce two squares to a rectangle; namely, that of reducing them to another square equal to their sum. The following solution was not difficult to find.

XVI.

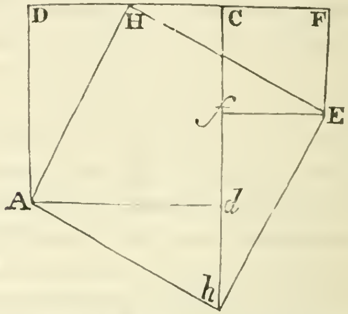
To make a square double of a given square.

Suppose in the first place that the two squares A B C D, C B F E, are equal to one another, and it is proposed to make a square equal to the two together. It may be easily observed that by drawing the diagonals A C and C F, the two triangles A B C and C B F, together will be equal to one of the squares. If therefore we make the triangles A B G, F B G, equal to the triangles D C A, C E F, we shall have the square A C F G, one side of which will be A C, the diagonal of the square A B C D, and which will enclose a surface equal to the two squares, as will be seen without the necessity of further demonstration.



XVII.

Let us propose to ourselves to make a square, equal to the two unequal squares A D C *d*, C F E *f*; or, what is the same, to change the figure A D F E *f d* into a square.



Following the spirit of the preceding method, we should inquire whether it be possible to find on the line D F, some point H, such, that

1. Drawing the lines A H, H E, and conceiving the triangles A D H, E F H, to turn round the points A and E, till they have the positions A *d h*, E *f h*, these two triangles meet at the point *h*.

2. That the four sides A H, H E, E *h*, *h* A, be equal and perpendicular to each other.

Now, such a point H, will be found by making D H equal to the side C F, or E F. For, it follows from the equality of D H and C F, in the first place, that if the triangle A D H turns round its angle A till it takes the position A *d h*, the point H having arrived at *h*, will be distant from the point C by an interval equal to D F.

From the same equality of D H and C F, it follows also that H F is equal to D C, and, therefore, when the triangle E F H, moving round E, to take the position E *f h*, shall have brought the point H to coincide with the point *h*, H or *h* will be from C, at a distance equal to D F.

The figure A D F E *f d* has consequently been changed into a figure of four sides A H E *h*. We have only to ascertain whether the four sides of this new figure are all equal and perpendicular to each other.

The equality of the four sides is evident, since A *h*, *h* E, are the lines A H, H E, and these are equal, as being corresponding sides of the triangles A D H, H F E, which are equal and similar, owing to the equality of D H, and C F or F E, of A D to H F, and the angles at D and F being right-angles.

Let us next examine whether the sides of the figure A H E *h*, meet at right-angles. But this may be easily proved; for, observe that while H A D turns round A, going to *h* A *d*, the side A H must move through the same space as the side A D. But the space which D A has to traverse to become A *d*, is measured by the angle D A *d*, (which is a right-angle) therefore the space which A H has to pass over to become A *h*, is measured also by a right-angle: consequently, A H and A *h* make a right-angle.

Hence it follows necessarily that the angles H, E, *h*, are right-angles: for, if a figure terminated by four equal sides, has one of its angles a right-angle, the other three must also be right-angles.

XVIII.

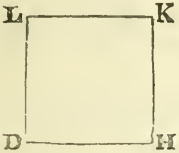
Whoever observes that the two squares A D C *d*, C F E *f*, are constructed, the one upon A D, one of the two smaller sides of the triangle A D H, and the other on E F, which is equal to D H, another of the smaller sides of the same triangle A D H; and that the square A H E *h*, equal to the two others taken together, is constructed on A H, the side commonly called the hypothenuse in a rectangular triangle; that side, namely, which is opposed to the right-angle; will have found out the famous property of rectangular triangles, namely, that the square of the hypothenuse is equal to the sum of the squares of the other two sides.

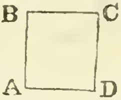
The greatest side of a rectangular triangle is called the hypothenuse.

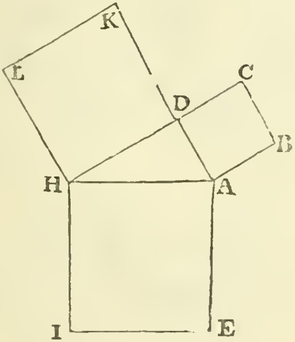
The square of the hypothenuse is equal to the sum of the squares of the two other sides.

XIX.

When therefore we wish to make one square equal to the two others H D K L, A B C D, it will not be necessary to place them side by side, and resolve them as we did in Art. XVII. All that is wanted is to make their sides A D, D H, form a right-angle ; then draw the line A H, which being the hypothenuse of the rectangular triangle A D H, must be the side of the square required.



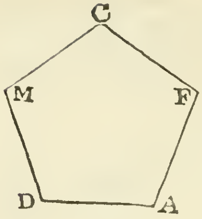


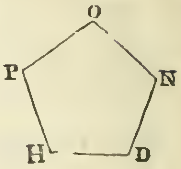


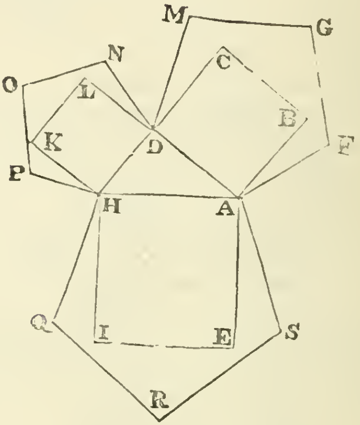
XX.

If we had two figures D A F G M, D H P O N, which we wished to reduce to one with a surface equal to that of the two taken together, we should only have to make a right-angle A D H, with the bases A D, H D, for its sides ; and the hypothenuse A H, of the triangle A D H, would be the base of the figure required.

If the sides of a rectangular triangle are made the bases of three similar figures, the figure constructed on the hypothenuse will be equal to the sum of the two others.







To be convinced of this, let us imagine the squares A B C D, D H K L, A H I E, constructed on the bases of the three similar figures. Then according to Art. XVIII. the square A H I E will be equal to the squares A B C D, D H K L, taken together. But similar figures are to each other as the squares of their homologous sides, (Part First, Art. XLVII.): therefore the three squares A B C D, D H K L, A H I E, will be the same proportional parts of the figures D A F G M, D H P O N, A H Q R S.

From this it will be easy to perceive that the figure A H Q R S will be equal to the other two together. For, suppose that each of these figures were the half of the figure which encloses it, no one would doubt that the figure A H Q R S was equal to the two others together, since its half would be equal to the halves of the other two together, namely, to the halves of D H P O N, D A F G M. But the inference is equally correct, when the squares A B C D, D H K L, A H I E, are the two-thirds, the three-fourths, &c., &c., of the figures D A F G M, D H P O N, A H Q R S.

XXI.

If we wished to join three, four, or more figures into one, or, what is the same, three, four, or more squares into one, the same method would answer our purpose. If the squares were three, after having joined two of them into one square, we might join the third to that, to which we had reduced the two others: and thus we should have a square equal to the three given squares.

XXII.

Hence it follows that if we wished to make a square five, six, or more times greater than a given one, the preceding method would suffice to resolve the problem. Even the inverse of that problem might be resolved by the same means: that is, we might make a square which should only be the fifth, sixth, or any part of a given square. This would only require that we should remember the method of finding a fourth proportional to three given lines. But in the third part of this work (Art. XXX.) the reader will find a more direct and convenient method of solving such problems.

XXIII.

The addition of similar figures furnishes a decisive proof of the inutility of graduated scales, for performing these operations, so that their truth shall be demonstrable.

Suppose that a person, ignorant of the preceding method, (Art. XVI.) wished to construct a square double of a given one, he would probably attempt it in the following manner:—

He would divide the side of the square into a great number of equal parts—100, for instance. Then multiplying 100 by 100, he would have 10,000 for the value of the square; and hence he would infer that the value of the required square, should be 20,000.

When a number is multiplied by itself, the product is called the square of that number.

But he could not deduce from this value the manner of describing the square; for this could not be obtained without having a number to express the side of the square, and which being multiplied by itself, that is, *squared*, should give for its product 20,000.

But such a number would be looked for in vain upon a scale, even if subdivisions were hundred parts of the side of the given square; for 141, multiplied by itself, gives 19,881, and 142 would give 20,164; products which either fall short of, or exceed the one required.

The person attempting this process might imagine that if he divided the side of the given square into smaller parts, so that they should be more than 100, he might succeed in finding a definite number of such parts, for the side of the square which was to be double of the given one. But he would be disappointed in these expectations, and would never be able to find two numbers, one of which expressed the side (that is, in the usual mathematical language, the *square root*) of a square, and the other the square root of a square double of the former.

The root of a square is the number which, multiplied by itself, gives the square.

XXIV.

The fact is, that it is demonstrated in Arithmetic, that if one of two numbers is not a multiple of the other; that is, if one of them does not contain the other exactly a certain number of times, the square of the greater will not be a multiple of the smaller. Thus 5, for instance, which cannot be *exactly* divided by 4, because four is not contained in it a certain number of times, will give a square, 25, which cannot be exactly divided by 16, the square of 4.

If therefore we square two numbers, of which one is greater than the other, and yet it is not quite the double of the smaller, we shall obtain two other numbers, one of which will be less than the quadruple of the other, without being either double or triple of it. We may therefore multiply in vain our subdivisions of the side of a square, for the side of a square twice as great (which, according to what has been demonstrated, Art. XVI., will be the diagonal of that square) will never be found to contain an exact number of those parts. Geometricians express this fact by saying, that the side of a square and its diagonal are incommensurable.

The side of a square and its diagonal are incommensurable.

XXV.

There are, however, many other lines which have no common measure between them.

For, let us write down the two series,

1, 2, 3, 4, 5, 6, 7, 8, 9, &c
1, 4, 9, 16, 25, 36, 49, 64, 81, &c.

The first of which expresses the natural numbers, and the other their squares; and we shall see that, as the numbers intervening between 4 and 9, between 9 and 16, and between 16 and 25, &c. have no square root, the sides of two squares, one of which shall be either triple, or quintuple, or sextuple, &c. of the other, must be incommensurable.

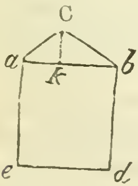
XXVI.

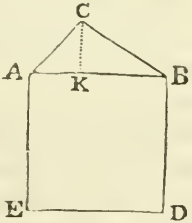
The fact that certain lines are incommensurable with other lines, might lead us to the suspicion that some of the propositions by means of which we have ascertained the proportionality of similar figures, were not perfectly true. In comparing such figures (Part First, Art. XXXIV. et seq.) we have proceeded on the supposition that there was a scale which might measure all their parts.

Now, however, it seems from what we have just said, that this view requires limitation. Let us therefore retrace our steps, and inquire whether those propositions should be modified.

XXVII.

In the first place, let us go back to what was said in Art. XXXIX. of Part First. Is it perfectly true that such triangles *a b c*, A B C, whose angles are the same, have their sides proportional to each other?





To answer this question, let us suppose that the base of the first being *a b*, that of the other is a right line A B, equal to the diagonal of a square having *a b* for its side, and upon this supposition, let us examine whether A C is to *a c* as A B is to *a b*.

It is unquestionably true that whatever be the number of parts into which we may arbitrarily suppose that *a b* may be divided, A B can never contain an exact number of those parts. But it is clear, nevertheless, that the greater we make the number of those parts, the more A B approaches the commensurability with *a b*. If we suppose *a b* divided into 100 parts, the number of such parts contained by A B will be between 141 and 142. (Art. XXIII.) Let us take up 141 and overlook the small difference. In that case we may say that A C contains 141 parts of *a c*.

Suppose in the next place, *a b* divided into 1000 parts, A B will then contain a number of those parts, between 1414 and 1415. Let us take up 1414 and neglect the remainder. It will also be found that A C contains 1414 of the one-thousand parts of *a c:* and generally, A C will contain as many parts of *a c*, with a remainder, as A B contains of *a b*, with a remainder.

As these remainders become smaller in regard to both lines, in proportion to the greatness of the number of parts into which we divide *a b*, there can be no doubt that if we imagine the division to proceed without limit, or *ad infinitum*, the remainders will disappear.

We may therefore conclude, that according to this view, the number of parts of *a c* contained by A C is equal to the number of parts of *a b* contained by A B: and therefore that A C is to *a c* as A B is to *a b*.

Similar triangles and other similar figures have their sides proportional, even when those sides are incommensurable.

It has been therefore rigorously demonstrated, that when the angles of two triangles are equal, their sides of the triangles are proportional, whether they have a common measure or not.

This reasoning proves the accuracy of the proposition contained in Art. XLV. of Part First, whence we reduce the proportionality of the corresponding sides of similar figures.

XXVIII.

Such figures are always as the squares of their homologous sides.

By a similar reasoning, we shall find that the propositions explained in Art. XLIV. and XLVII. of Part First, those, namely, where we showed that the areas of triangles, and other figures, when similar, have between them the same proportion as the squares of their homologous sides—are universally true, even when their sides have no common measure.

Let us, for instance, consider the similar triangles (in the preceding figure) A B C, *a b c*, whose altitudes we suppose to be incommensurable. In such a case there can be no square, if ever so small, which shall measure both triangles, and the square constructed on their bases. This is to say, that the areas *a b c* and *a b d e* are incommensurable with each other. The areas A B C and A B D E will be also incommensurable. Nevertheless the triangle A B C will be to the square A B D E as the triangle abc to the square $abde$.

The certainty of this will appear more and more, if we observe that in proportion to the smallness of the division of the scale by which we may measure the lines A B and C K, we shall come nearer to the numbers which would express the ratio of A B C to A B D E. By dividing therefore the scale of the triangle abc into the same number of parts, and neglecting the remainder, we shall find that the same numbers will express the ratio of the triangle A B C to the square A B D E, and that of the triangle abc to the square $abde$. If we carry on by thought, this division of the scale into smaller and smaller parts without end, the remainder will disappear; and upon the truth of this unquestionable notion, we may assert, that, were there a number which would express the ratio of the triangle abc to the square $abde$, that number would also express the ratio of the triangle A B C to the square A B D E. Whence we conclude, that the triangle A B C is to the square A B D E as the triangle abc is to the square $abde$.

The same must be asserted of all similar figures.

END OF THE SECOND PART.

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